Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208 Solutions to the Specimen Examination Paper

This is a guide to the type of written solutions required. We do not expect your solutions to be as neatly laid out as these; and, for some questions, there are alternative ways of doing them. Any correct method receives full marks unless the question specifically asks for a particular method.

A mark scheme is provided so that you can mark your own attempts. This uses accuracy marks (A-marks) and method marks (M-marks) with an indication as to how these are awarded.

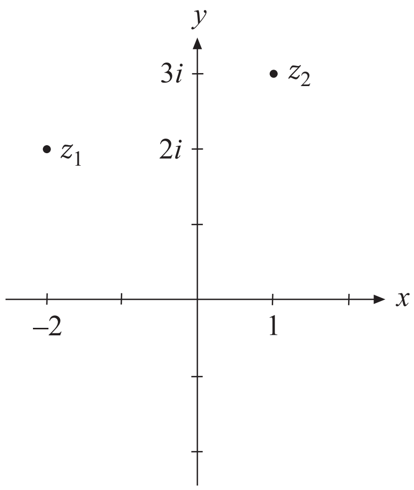
A few comments are included to try to help you with your revision and your examination technique.

SOLUTIONS TO SECTION 1

Mark scheme

Question 1

(a)



1A ($\frac{1}{2}$ for each)

(b) $|z_1| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$. — 1A for $|z_1|$

As z_1 is in the second quadrant and the line from 0 to z_1 makes an angle of $\pi/4$ with the negative real axis, $\operatorname{Arg} z_1 = 3\pi/4$. — 1A for $\operatorname{Arg} z_1$

(c) $\dfrac{z_1}{z_2} = \dfrac{-2+2i}{1+3i} = \dfrac{(-2+2i)(1-3i)}{(1+3i)(1-3i)} = \dfrac{4+8i}{10} = \dfrac{2+4i}{5}$ (or $\dfrac{2}{5} + \dfrac{4}{5}i$). — 1M, 1A

There are many equivalent ways of writing the final answer – the mark would be given for any correct version, including $\dfrac{4+8i}{10}$.

5 Total

Question 2

We first show that A is a subset of B.

Let $(x, y) \in A$. Then

$$x^2 + y^2 \leq 4.$$

It follows that

$$x^2 \leq 4$$

and hence that

$$x \leq 2.$$

So $(x, y) \in B$. 2 argument

Thus A is a subset of B.

We now show that A is a **proper** subset of B, that is, we show that there is a point in B that is not in A.

The point $(0, 3) \in B$, since 1 suitable point

$$0 \leq 2,$$ 1 showing in B

but $(0, 3) \notin A$ since

$$0^2 + 3^2 = 9 \not\leq 4.$$ 1 showing not in A

So $A \neq B$, and hence A is a proper subset of B.

There are many possible choices here, namely any point (x, y) for which $x \leq 2$ but $x^2 + y^2 > 4$.

5 Total

Question 3

(a) $\left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 2 & 3 & -1 & 3 \\ 1 & 4 & -2 & 2 \end{array}\right)$ 1A

(b)

$$\begin{array}{ll}
\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} & \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 2 & 3 & -1 & 3 \\ 1 & 4 & -2 & 2 \end{array}\right) \\[2ex]
\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} & \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 5 & -3 & 1 \\ 0 & 5 & -3 & 1 \end{array}\right) \\[2ex]
\begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 \end{array} & \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\[2ex]
\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{5}\mathbf{r}_2 \\ \\ \end{array} & \left(\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{array}\right) \\[2ex]
\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_2 \\ \\ \\ \end{array} & \left(\begin{array}{rrr|r} 1 & 0 & \frac{2}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)
\end{array}$$

1M

1A for calculations

1A row-reduced

The last matrix given is the row-reduced form.

Remember that you need to get the matrix in row-reduced form and not just to a state where you can solve the equations.

(c) The row-reduced matrix gives the equations

$$\begin{aligned} x \quad + \tfrac{2}{5}z &= \tfrac{6}{5} \\ y - \tfrac{3}{5}z &= \tfrac{1}{5} \end{aligned}$$ 1A for equations

so, taking $z = k$, the general solution is

$$\begin{aligned} x &= \tfrac{6}{5} - \tfrac{2}{5}k \\ y &= \tfrac{1}{5} + \tfrac{3}{5}k \\ z &= k, \quad \text{where } k \in \mathbb{R}. \end{aligned}$$ 1A for general solution

If you have only two equations for three unknowns, then remember to set $z = k$.

6 Total

Question 4

(a) The standard basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$.

$$t(1,0) = (3,2) \quad \text{and} \quad t(0,1) = (-4,1).$$

Hence the matrix is

$$\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}.$$ 1A

(b) $t(2,1) = (2,5)$ and $t(1,1) = (-1,3)$.

Hence the matrix is

$$\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}.$$ 1M 1A

(c) We have $t(2,1) = (2,5)$. Let $(2,5) = a(2,1) + b(1,1)$.

Thus

$$\begin{aligned} 2a + b &= 2, \\ a + b &= 5. \end{aligned}$$ $\frac{1}{2}$ M

So $a = -3$, $b = 8$. $\frac{1}{2}$ A

Also, $t(1,1) = (-1,3)$. Let $(-1,3) = c(2,1) + d(1,1)$.

Thus

$$\begin{aligned} 2c + d &= -1, \\ c + d &= 3. \end{aligned}$$ $\frac{1}{2}$ M

So $c = -4$ and $d = 7$. $\frac{1}{2}$ A

Hence the matrix is

$$\begin{pmatrix} -3 & -4 \\ 8 & 7 \end{pmatrix}.$$ 1A

6 Total

Question 5

(a) The elements of G are as follows:

e	identity symmetry	
$(1\,4)(2\,5)(3\,6)$	rotation through π about the centre	
$(1\,5)(2\,4)$	reflection in the horizontal axis	3A cycle forms
$(1\ 2)(3\ 6)(4\ 5)$	reflection in the vertical axis	2A descriptions

(b) Choose a permutation, say k, and use the renaming method to find the conjugate subgroup $k \circ G \circ k^{-1}$. (If $k \circ G \circ k^{-1} = G$, then try a different permutation k.) There are many possible answers here.

Let $k = (1\ 2)$. Then we can take

$$H = k \circ G \circ k^{-1} = \{e,\ (2\ 4)(1\ 5)(3\ 6),\ (2\ 5)(1\ 4),\ (2\ 1)(3\ 6)(4\ 5)\},$$ 1M, 1A

with conjugating permutation k. 1A

An alternative way to find a conjugate subgroup H is to rearrange the vertex labels on the figure and then write down the elements of the symmetry group of the resulting new labelled figure, in cycle form. The permutation that specifies how the labels have been rearranged will conjugate G to H.

8 Total

Question 6

(a) To find the identity element of G, we systematically compose pairs of numbers from G until we find a pair x, y such that $x \times_{18} y = y$; then x is the identity element.

We have $10 \times_{18} 2 = 2$, so the identity element of G is 10. 1M, 1A

(b) Usually the easiest way to show that a set of order 2 is a subgroup is to demonstrate that it is the subgroup generated by some element. Another approach is to check the three subgroup properties, but this takes longer.

We have

$$8 \times_{18} 8 = 10,$$ 1A

and 10 is the identity element, so 8 has order 2 in G and hence

$$\langle 8 \rangle = \{10, 8\} = H.$$

That is, H is the cyclic subgroup of G generated by 8. 1M

(c) H is a normal subgroup of G because G is abelian. 1

The cosets of H in G are

$$H = \{10, 8\}$$
$$2H = \{2, 16\}$$
$$4H = \{4, 14\}.$$ 1M, 1A

These cosets are calculated as follows:
$2H = \{2 \times_{18} 10, 2 \times_{18} 8\} = \{2, 16\}$ and
$4H = \{4 \times_{18} 10, 4 \times_{18} 8\} = \{4, 14\}$.

(d) $G/H \cong C_3$. 1

8 Total

Question 7

(a) We have to prove that, for all $\mathbf{A}, \mathbf{B}$ in G,

$$\phi(\mathbf{A}+\mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B}).$$

Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ be in G. $\frac{1}{2}$M

Then

$$\phi(\mathbf{A}+\mathbf{B}) = \phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right)$$

$$= \phi\begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} = a+c+b+d. \qquad \tfrac{1}{2}\text{A}$$

Also

$$\phi(\mathbf{A}) + \phi(\mathbf{B}) = \phi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \phi\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = (a+b)+(c+d)$$

$$= a+c+b+d \qquad \tfrac{1}{2}\text{A}$$

$$= \phi(\mathbf{A}+\mathbf{B}). \qquad \tfrac{1}{2}\text{M}$$

So ϕ is a homomorphism.

(b) The identity element of the codomain group is 0.

$$\operatorname{Ker}\phi = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : \phi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 0\right\} \qquad 1\text{M}$$

$$= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a+b = 0\right\}$$

$$= \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R}\right\}. \qquad 1\text{A}$$

We guess that $\operatorname{Im}\phi = \mathbb{R}$. To confirm this, we show that $\operatorname{Im}\phi \subseteq \mathbb{R}$ and $\mathbb{R} \subseteq \operatorname{Im}\phi$.

We know that $\operatorname{Im}\phi \subseteq \mathbb{R}$. $\frac{1}{2}$A

Also, for any $r \in \mathbb{R}$, $\phi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = r$ and $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in G$, $\frac{1}{2}$M

so $\mathbb{R} \subseteq \operatorname{Im}\phi$. $\frac{1}{2}$A

Hence $\operatorname{Im}\phi = \mathbb{R}$. $\frac{1}{2}$A

An alternative way to show that $\operatorname{Im}\phi = \mathbb{R}$ is as follows.

$$\operatorname{Im}\phi = \left\{\phi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G\right\} \qquad 1\text{M}$$

$$= \{a+b : a, b \in \mathbb{R}\} \qquad \tfrac{1}{2}\text{A}$$

$$= \mathbb{R}. \qquad \tfrac{1}{2}\text{A}$$

(c) By the First Isomorphism Theorem, 1M

$$G/\operatorname{Ker}\phi \cong \operatorname{Im}\phi = (\mathbb{R}, +). \qquad 1\text{A}$$

8 Total

Question 8

(a) For large n, a_n is approximately $\dfrac{1}{2n}$, so we use the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

Let $a_n = \dfrac{2n-1}{4n^2+1}$, $\quad b_n = \dfrac{1}{n}$. $\quad \frac{1}{2}$A suitable choice of b_n

First you need to remember to check that the condition of the Limit Comparison Test is satisfied.

Then $a_n > 0$, $b_n > 0$ and $\quad \frac{1}{2}$M

$$\frac{a_n}{b_n} = \frac{2n^2-n}{4n^2+1} = \frac{2-1/n}{4+1/n^2} \to \frac{1}{2} \neq 0 \text{ as } n \to \infty. \qquad \tfrac{1}{2}\text{M, 1A}$$

This follows from the Combination Rules for sequences since $\left(\dfrac{1}{n}\right)$ and $\left(\dfrac{1}{n^2}\right)$ are basic null sequences. You do not need to say this in a question on series, although you would in a question on sequences.

Now

$$\sum_{n=1}^{\infty} b_n \text{ is divergent} \qquad \tfrac{1}{2}\text{A}$$

and so

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \qquad \tfrac{1}{2}\text{A}$$

by the Limit Comparison Test. $\quad \frac{1}{2}$M

(b) Let $a_n = \dfrac{n^2 e^n}{(n+1)!}$. $\quad \frac{1}{2}$M

We consider the Ratio Test because of the factorial term.

Then $a_n > 0$ and $\quad \frac{1}{2}$M

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 e^{n+1}}{(n+2)!} \frac{(n+1)!}{n^2 e^n} \qquad \tfrac{1}{2}\text{M, } \tfrac{1}{2}\text{A}$$

$$= \left(\frac{n+1}{n}\right)^2 \frac{e}{n+2} = \left(1+\frac{1}{n}\right)^2 \frac{e}{n+2} \to 0 \text{ as } n \to \infty. \qquad \text{1A}$$

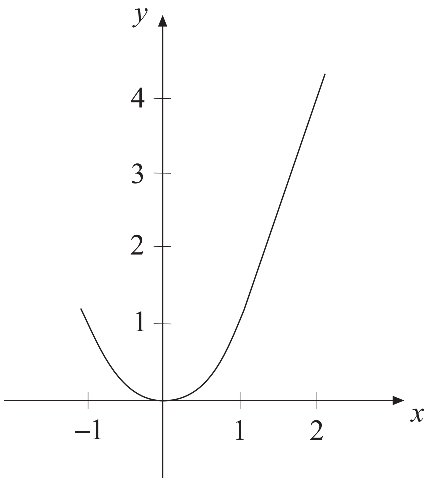
So by the Ratio Test, $\quad \frac{1}{2}$M

$$\sum_{n=1}^{\infty} a_n \text{ is convergent.} \qquad \tfrac{1}{2}\text{A}$$

8 Total

Question 9

(a)



$\frac{1}{2}$A shape of x^2
$\frac{1}{2}$A shape of $3x - 2$

(b) The function f is continuous at 1. For, let $\frac{1}{2}$A

$$g(x) = x^2, \; h(x) = 3x - 2 \text{ and } I = (0, 2).$$

$\frac{1}{2}$A functions
$\frac{1}{2}$A I

I can be any open interval containing 1.

Then

1. $f(x) = \begin{cases} g(x), & x \in I, x < 1, \\ h(x), & x \in I, x > 1. \end{cases}$ $\frac{1}{2}$M

2. $f(1) = g(1) = h(1) = 1.$ 1A

3. g and h are continuous at 1 (basic continuous functions.) $\frac{1}{2}$M

It follows that f is continuous at 1, by the Glue Rule for continuous functions. $\frac{1}{2}$M Glue Rule

(c) The function f is not differentiable at 1. $\frac{1}{2}$A

We apply the Glue Rule for differentiable functions. $\frac{1}{2}$M

The first few steps of the Glue Rule for differentiable functions are the same as for the Glue Rule for continuous functions, so we do not need to repeat them here.

In addition to the properties checked in part (b), we have g and h are differentiable at 1. $\frac{1}{2}$M

Since $g'(x) = 2x$ and $h'(x) = 3$, we have $\frac{1}{2}$A

$$g'(1) = 2 \neq 3 = h'(1).$$ 1A

8 Total

Question 10

(a) Here

$$f(x) = \frac{x}{x-1} \qquad f(2) = 2; \qquad \tfrac{1}{2}\mathsf{A}$$

$$f'(x) = \frac{-1}{(x-1)^2}, \qquad f'(2) = -1; \qquad 1\mathsf{A},\ \tfrac{1}{2}\mathsf{A}$$

$$f''(x) = \frac{2}{(x-1)^3}, \qquad f''(2) = 2. \qquad \tfrac{1}{2}\mathsf{A},\ \tfrac{1}{2}\mathsf{A}$$

So

$$\begin{aligned} T_1(x) &= f(2) + f'(2)(x-2) && \tfrac{1}{2}\mathsf{M} \\ &= 2 - (x-2), && \tfrac{1}{2}\mathsf{A} \\ T_2(x) &= T_1(x) + \frac{f''(2)}{2!}(x-2)^2 && \tfrac{1}{2}\mathsf{M} \\ &= 2 - (x-2) + (x-2)^2. && \tfrac{1}{2}\mathsf{A} \end{aligned}$$

(b) Now we use Strategy F12 with $I = [2, 2.25]$, $a = 2$, $r = 0.25$ and $n = 2$. $\tfrac{1}{2}$M

1. First, $f'''(x) = \dfrac{-6}{(x-1)^4}$. $\tfrac{1}{2}$A

2. Thus $|f'''(c)| = \dfrac{6}{(c-1)^4} \leq \dfrac{6}{(2-1)^4} = 6$, for $c \in [2, 2.25]$, $\tfrac{1}{2}$A

   so we can take $M = 6$.

3. Hence

$$\begin{aligned} |R_2(x)| &\leq \frac{M}{(2+1)!} r^{2+1} && \tfrac{1}{2}\mathsf{M} \\ &= \frac{6}{3!} \times 0.25^3 && \tfrac{1}{2}\mathsf{A} \\ &= \left(\frac{1}{4}\right)^3 = \frac{1}{64}, \quad \text{for } x \in [2, 2.25]. && \tfrac{1}{2}\mathsf{A} \end{aligned}$$

8 Total

SOLUTIONS TO SECTION 2

Question 11

(a) Putting $a = b = 0$ gives $0 \in S$. $\frac{1}{2}$A

Let

$$\mathbf{v}_1 = (a_1 + b_1, -a_1 + 2b_1, 2a_1 - b_1) \in S$$
$$\mathbf{v}_2 = (a_2 + b_2, -a_2 + 2b_2, 2a_2 - b_2) \in S. \quad \tfrac{1}{2}\text{M}$$

Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (a_1 + b_1 + a_2 + b_2, -a_1 + 2b_1 - a_2 + 2b_2, 2a_1 - b_1 + 2a_2 - b_2)\\ &= (a_1 + a_2 + b_1 + b_2, -(a_1 + a_2) + 2(b_1 + b_2), 2(a_1 + a_2) - (b_1 + b_2)) \in S.\end{aligned}$$

1A

So S is closed under vector addition. $\frac{1}{2}$M

Let $\mathbf{v} = (a + b, -a + 2b, 2a - b) \in S, \alpha \in \mathbb{R}$. $\frac{1}{2}$M

$$\begin{aligned}\text{Then } \alpha\mathbf{v} &= \alpha(a + b, -a + 2b, 2a - b)\\ &= (\alpha(a + b), \alpha(-a + 2b), \alpha(2a - b)))\\ &= (\alpha a + \alpha b, -\alpha a + 2\alpha b, 2\alpha a - \alpha b) \in S.\end{aligned}$$ 1A

So S is closed under scalar multiplication. $\frac{1}{2}$M

So S is a subspace of $\mathbb{R}^3$. $\frac{1}{2}$M

(b) First, $(1, -1, 2)$ and $(1, 2, -1)$ both belong to S (taking $a = 1$, $b = 0$ and $a = 0$, $b = 1$ respectively). 1A

Let $\mathbf{v} = (a + b, -a + 2b, 2a - b) \in S$. Then

$$\mathbf{v} = a(1, -1, 2) + b(1, 2, -1).$$ $\frac{1}{2}$A

So $\{(1, -1, 2), (1, 2, -1)\}$ spans S. $\frac{1}{2}$M

Also $\{(1, -1, 2), (1, 2, -1)\}$ is linearly independent and so is a basis $\frac{1}{2}$M
for S. $\frac{1}{2}$A

The dimension of S is 2. 1A

(c) Using Gram–Schmidt with $\mathbf{w}_1 = (1, -1, 2)$ and $\mathbf{w}_2 = (1, 2, -1)$ gives $\frac{1}{2}$M

$$\mathbf{v}_1 = \mathbf{w}_1 = (1, -1, 2),$$ $\frac{1}{2}$A

$$\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\mathbf{v}_1.\mathbf{w}_2}{\mathbf{v}_1.\mathbf{v}_1}\right)\mathbf{v}_1$$ $\frac{1}{2}$M

$$= (1, 2, -1) - \frac{(1, -1, 2).(1, 2, -1)}{(1, -1, 2).(1, -1, 2)}(1, -1, 2)$$

$$= (1, 2, -1) + \frac{3}{6}(1, -1, 2)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, 0\right).$$ 1A

So an orthogonal basis is $\left\{(1, -1, 2), \left(\frac{3}{2}, \frac{3}{2}, 0\right)\right\}$. $\frac{1}{2}$A

(d) $$(2,1,1) = \frac{(2,1,1).(1,-1,2)}{(1,-1,2).(1,-1,2)}(1,-1,2) + \frac{(2,1,1).\left(\frac{3}{2},\frac{3}{2},0\right)}{\left(\frac{3}{2},\frac{3}{2},0\right)\cdot\left(\frac{3}{2},\frac{3}{2},0\right)}\left(\frac{3}{2},\frac{3}{2},0\right)$$

$$= \frac{3}{6}(1,-1,2) + \frac{9/2}{9/2}\left(\frac{3}{2},\frac{3}{2},0\right)$$ 1M

$$= \frac{1}{2}(1,-1,2) + \left(\frac{3}{2},\frac{3}{2},0\right)$$ 1A

So the coordinates are $\left(\frac{1}{2},1\right)$. 1A

15 Total

Question 12

(a) p has order 6 and is even. $\frac{1}{2}$A, $\frac{1}{2}$A

(b) (i) There is no need to use the usual method for composing permutations. Instead, to calculate p^2, we use the fact that p^2 maps each symbol *two places* around the cycle of p in which the symbol lies. Thus, for example, p^2 maps $1 \to 3 \to 5$ and $7 \to 7$. Similarly p^3 maps each symbol *three places* around the cycle of p in which it lies, and so on.

$$\langle p \rangle = \{e,\ (1\ 2\ 3\ 4\ 5\ 6)(7\ 8),\ (1\ 3\ 5)(2\ 4\ 6),\ (1\ 4)(2\ 5)(3\ 6)(7\ 8),\ (1\ 5\ 3)(2\ 6\ 4),\ (1\ 6\ 5\ 4\ 3\ 2)(7\ 8)\}$$

3A ($\frac{1}{2}$ per element)

(ii) Find the cyclic subgroup generated by each element of $\langle p \rangle$. Some elements will generate the same cyclic subgroup.

The cyclic subgroups of $\langle p \rangle$ are $\frac{1}{2}$M

$\langle e \rangle = \{e\}$, $\frac{1}{2}$A

$\langle (1\ 4)(2\ 5)(3\ 6)(7\ 8) \rangle = \{e,\ (1\ 4)(2\ 5)(3\ 6)(7\ 8)\}$ 1A

$\langle (1\ 3\ 5)(2\ 4\ 6) \rangle = \langle (1\ 5\ 3)(2\ 6\ 4) \rangle = \{e,\ (1\ 3\ 5)(2\ 4\ 6),\ (1\ 5\ 3)(2\ 6\ 4)\}$ 1A

$\langle (1\ 2\ 3\ 4\ 5\ 6)(7\ 8) \rangle = \langle (1\ 6\ 5\ 4\ 3\ 2)(7\ 8) \rangle = \langle p \rangle$ 1A

Since $\langle p \rangle$ is a cyclic group all its subgroups are cyclic and hence it has no further subgroups. 1M

An alternative to finding the cyclic subgroup generated by *every* element of $\langle p \rangle$ is to state that since $\langle p \rangle$ is a cyclic group of order 6 it has exactly one cyclic subgroup of order q for each positive divisor q of 6 and no other subgroups, and find a cyclic subgroup of each of the orders 1, 2, 3 and 6.

(c) (i) Since p lies in A_8, the whole of the subgroup generated by p must also lie in A_8, since A_8 is a group.

Since p is even, it is an element of A_8 and hence $\langle p \rangle$ is a subgroup of A_8. 1M

(ii) Two such elements are $(1\ 2\ 3\ 4\ 5\ 6)$ and $(1\ 2\ 3)(4\ 5)$. 2A

(Two others are $(1\ 2\ 3)(4\ 5\ 6)(7\ 8)$ and $(1\ 2\ 3)(4\ 5)(6\ 7)$.)

(iii) The subgroup generated by an *odd* element of order 6 in S_6 will have the required properties.

Such a subgroup is

$$\langle (1\ 2\ 3\ 4\ 5\ 6) \rangle = \{e,\ (1\ 2\ 3\ 4\ 5\ 6),\ (1\ 3\ 5)(2\ 4\ 6),\ (1\ 4)(2\ 5)(3\ 6),\ (1\ 5\ 3)(2\ 6\ 4),\ (1\ 6\ 5\ 4\ 3\ 2)\}$$

1M, 1A

This subgroup contains the permutation $(1\ 2\ 3\ 4\ 5\ 6)$, which is odd, because it is a cycle of even length. Therefore this subgroup is not a subgroup of A_8. 1M

15 Total

Question 13

(a) (i) Dividing through by the dominant term $n!$, we obtain

$$a_n = \frac{3^n/n! + 1}{1 + 2^n/n!}.$$ ½A

Since $(3^n/n!)$ and $(2^n/n!)$ are basic null sequences, we deduce by ½M
the Combination Rules that ½M

Don't forget to identify the basic null sequences and to refer to the Combination Rules in a question on sequences.

$$\lim_{n\to\infty} a_n = \frac{0+1}{1+0} = 1.$$ 1A

(ii) The dominant term is 3^n.

Since this is in the numerator, we consider $1/a_n$ and divide through by the dominant term.

$$\frac{1}{a_n} = \frac{2n + 2^n - 3}{3^n + n + 1}$$ ½A

$$= \frac{2n/3^n + (2/3)^n - 3/3^n}{1 + n/3^n + 1/3^n}$$ ½A

Since $(n/3^n)$, $((2/3)^n)$ and $(1/3^n)$ are basic null sequences, we ½M
deduce by the Combination Rules that ½M

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0+0-0}{1+0+0} = 0.$$ ½A

We have

1. $\left(\dfrac{1}{a_n}\right)$ is a null sequence;
2. a_n is positive for each n. ½M

Thus, by the Reciprocal Rule, (a_n) tends to infinity, so ½M

(a_n) is divergent. ½A

(iii) We consider two subsequences (the odd and even subsequences). This is because the sequence takes positive values when n is even and negative values when n is odd.

First consider

$$a_{2k} = \frac{(-1)^{2k}2k}{4k+1}$$

$$= \frac{2}{4 + 1/k}.$$ ½A

Since $(1/k)$ is a basic null sequence, we deduce by the ½M
Combination Rules that ½M

$$\lim_{k\to\infty} a_{2k} = \frac{2}{4+0} = \frac{1}{2}.$$ ½A

Now consider

$$a_{2k+1} = \frac{(-1)^{2k+1}(2k+1)}{2(2k+1)+1}$$

$$= \frac{-2k-1}{4k+3}$$

$$= \frac{-2 - 1/k}{4 + 3/k}.$$ ½A

Since $(1/k)$ is a basic null sequence, we deduce by the Combination Rules that

$$\lim_{k\to\infty} a_{2k+1} = \frac{-2-0}{4+0} = -\frac{1}{2}.$$ $\frac{1}{2}$A

Thus (a_n) has two convergent subsequences with different limits. $\frac{1}{2}$M

So, by the First Subsequence Rule, $\frac{1}{2}$M

(a_n) is divergent. $\frac{1}{2}$A

(b) We guess that 1 is the least upper bound of E. To show this, we use Strategy D1.

1. $1 - \frac{2}{n^2} < 1$, for all $n = 1, 2, \ldots$. 1A

This shows that 1 is an upper bound. We now show that every $M' < 1$ is not an upper bound.

2. Let $M' < 1$. $\frac{1}{2}$M

Then

$$1 - \frac{2}{n^2} > M'$$

$$\iff 1 - M' > \frac{2}{n^2}$$ $\frac{1}{2}$M

$$\iff n^2 > \frac{2}{1-M'}, \quad \text{since } 1 - M' > 0 \text{ and } n^2 > 0,$$

$$\iff n > \sqrt{\frac{2}{1-M'}}, \quad \text{since } n > 0.$$ $\frac{1}{2}$A

By the Archimedean Property of $\mathbb{R}$, there exists a positive integer n $\frac{1}{2}$M

such that $n > \sqrt{\frac{2}{1-M'}}$ and hence $1 - \frac{2}{n^2} > M'$. $\frac{1}{2}$A

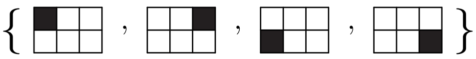
Thus 1 is the least upper bound of E. $\frac{1}{2}$A

15 Total

SOLUTIONS TO SECTION 3

Question 14

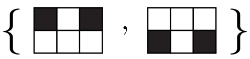
(a) The first figure has orbit



2A

and stabiliser $\{e\}$. 1A

The second figure has orbit



1A

and stabiliser $\{e, r\}$. 1A

The Orbit–Stabiliser Theorem provides a useful check here.

(b) (i) We can find the sizes of the fixed sets either with or without using the permutation method. Here we use the permutation method.

We can label the regions as follows.

1	2	3
4	5	6

We obtain the following.

Symmetry g	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
e	$(1)(2)(3)(4)(5)(6)$	6	2^6
a	$(1\ 6)(2\ 5)(3\ 4)$	3	2^3
r	$(1\ 3)(2)(4\ 6)(5)$	4	2^4
s	$(1\ 4)(2\ 5)(3\ 6)$	3	2^3

3M, 2A

(ii) The number required is the number of orbits of the group action. By the Counting Theorem, this is

$$\begin{aligned}\tfrac{1}{4}\left(2^6 + 2 \times 2^3 + 2^4\right) &= \tfrac{1}{4} \times 2^4\left(2^2 + 2\right)\\ &= 4 \times 6\\ &= 24.\end{aligned}$$

1M, 1A

(c) We use the group action axioms and the group table of $S(\square)$.

$$\begin{aligned}r \wedge A &= (a \circ s) \wedge A\\ &= a \wedge (s \wedge A)\\ &= a \wedge A\\ &= B.\end{aligned}$$

3M

15 Total

Question 15

(a) (i) Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$ such that

$$|f(x) - f(2)| < \varepsilon \text{ for all } x \text{ with } |x-2| < \delta.$$ 1M

We rewrite the expression for $|f(x) - f(2)|$ to obtain an expression including $|x-2|$ since we want to use the fact that $|x-2| < \delta$.

Now

$$\begin{aligned}|f(x) - f(2)| &= |(x^2 - 3x) - (-2)|\\ &= |x^2 - 3x + 2|\\ &= |(x-2)(x-1)|.\end{aligned}$$ 1A

If $|x-2| \le 1$ then x lies in $[1,3]$, so $\tfrac{1}{2}$A

$$\begin{aligned}|x-1| &\le |x| + |1| \quad \text{(by the Triangle Inequality)}\\ &\le 3 + 1 = 4\end{aligned}$$ $\tfrac{1}{2}$A

and hence

$$|f(x) - f(2)| \le 4|x-2|.$$ $\tfrac{1}{2}$A

So if we choose $\delta = \min\{1, \varepsilon/4\}$, then $\tfrac{1}{2}$M, $\tfrac{1}{2}$A

$$|x-2| < \delta \Longrightarrow |f(x) - f(2)| < 4\delta \le \varepsilon.$$ $\tfrac{1}{2}$A

So f is continuous at 2.

(ii) The function $f(x) = \dfrac{2x}{x^2-3}$ is a rational function, and so is continuous on its domain (which is $\mathbb{R} - \{\pm\sqrt{3}\}$); in particular, it is $\tfrac{1}{2}$M continuous on domain including $[-1,1]$ domain

continuous on the interval $[-1,1]$. $\tfrac{1}{2}$A

Since this interval is a bounded closed interval, it follows that f is $\tfrac{1}{2}$M
uniformly continuous on $[-1,1]$ $\tfrac{1}{2}$A

(b) Let $f(x) = \sin(x^2)$ and $g(x) = 2e^x - e^{2x} - 1$. $\tfrac{1}{2}$M

Don't forget to check the conditions for l'Hôpital's Rule.

Then f and g are differentiable on $\mathbb{R}$ and $f(0) = g(0) = 0$. $\tfrac{1}{2}$M, $\tfrac{1}{2}$A

So by l'Hôpital's Rule $\tfrac{1}{2}$M

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$ $\tfrac{1}{2}$M

if this limit exists.

Now

$$f'(x) = 2x\cos(x^2) \text{ and } g'(x) = 2e^x - 2e^{2x}$$ 1A

and hence

$$f'(0) = 0 \text{ and } g'(0) = 0.$$ $\tfrac{1}{2}$A

Both f' and g' are differentiable on $\mathbb{R}$ so, by l'Hôpital's Rule, $\tfrac{1}{2}$M

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)}$$ $\tfrac{1}{2}$M

if this limit exists.

Now

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2) \text{ and } g''(x) = 2e^x - 4e^{2x}$$ 1A

and hence

$$f''(0) = 2 \text{ and } g''(0) = -2.$$ $\frac{1}{2}$A

Both f'' and g'' are continuous functions, so $\frac{1}{2}$M

Don't forget to justify the statement below by stating that f'' and g'' are continuous.

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \frac{f''(0)}{g''(0)} = \frac{2}{-2} = -1.$$ $\frac{1}{2}$A

So $\lim_{x\to 0}\frac{f(x)}{g(x)} = -1.$ $\frac{1}{2}$A

Here we work backwards and first deduce that

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = -1$$

and then deduce that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = -1.$$

You do not need to write this.

15 Total